Russell's mathematical logic

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Mathematical logic, which is nothing else but a precise and complete formulation of formal logic, has two quite different aspects. On the one hand, it is a section of Mathematics treating of classes, relations, combinations of symbols, etc., instead of numbers, functions, geometric figures, etc. On the other hand, it is a science prior to all others, which contains the ideas and principles underlying all sciences. It was in this second sense that Mathematical Logic was first conceived by Leibniz in his *Characteristica universalis,* of which it would have formed a central part. But it was almost two centuries after his death before his idea of a logical calculus really sufficient for the kind of reasoning occurring in the exact sciences was put into effect (in some form at least, if not the one Leibniz had in mind) by Frege and Peano.[1] Frege was chiefly interested in the analysis of thought and used his calculus in the first place for deriving arithmetic from pure logic. Peano, on the other hand, was more interested in its applications within mathematics and created an elegant and flexible symbolism, which permits expressing even the most complicated mathematical theorems in a perfectly precise and often very concise manner by single formulas.

It was in this line of thought of Frege and Peano that Russell's work set in. Frege, in consequence of his painstaking analysis of the proofs, had not gotten beyond the most elementary properties of the series of integers, while Peano had accomplished a big collection of mathematical theorems expressed in the new symbolism, but without proofs. It was

Reprinted with the kind permission of the author, editor, and publisher from Paul A. Schilpp, ed., *The Philosophy of Bertrand Russell,* The Library of Living Philosophers, Evanston, Ill. (Evanston & Chicago: Northwestern University, 1944), pp. 125–53. The author asked to note (1) that since the original publication of this paper advances have been made in some of the problems discussed and that the formulations given could be improved in several places, and (2) that the term "constructivistic" in this paper is used for a strictly anti-realistic kind of constructivism. Its meaning, therefore, is not identical with that used in current discussions on the foundations of mathematics. If applied to the actual development of logic and mathematics it is equivalent with a certain kind of "predicativity" and hence different both from "intuitionistically admissible" and from "constructive" in the sense of the Hilbert School.

[1]Frege has doubtless the priority, since his first publication about the subject, which already contains all the essentials, appeared ten years before Peano's.

only in *Principia Mathematica* that full use was made of the new method for actually deriving large parts of mathematics from a very few logical concepts and axioms. In addition, the young science was enriched by a new instrument, the abstract theory of relations. The calculus of relations had been developed before by Peirce and Schröder, but only with certain restrictions and in too close analogy with the algebra of numbers. In *Principia* not only Cantor's set theory but also ordinary arithmetic and the theory of measurement are treated from this abstract relational standpoint.

It is to be regretted that this first comprehensive and thorough going presentation of a mathematical logic and the derivation of Mathematics from it is so greatly lacking in formal precision in the foundations (contained in *1–*21 of *Principia*), that it presents in this respect a considerable step backwards as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism. Syntactical considerations are omitted even in cases where they are necessary for the cogency of the proofs, in particular in connection with the "incomplete symbols." These are introduced not by explicit definitions, but by rules describing how sentences containing them are to be translated into sentences not containing them. In order to be sure, however, that (or for what expressions) this translation is possible and uniquely determined and that (or to what extent) the rules of inference apply also to the new kind of expressions, it is necessary to have a survey of all possible expressions, and this can be furnished only by syntactical considerations. The matter is especially doubtful for the rule of substitution and of replacing defined symbols by their *definiens*. If this latter rule is applied to expressions containing other defined symbols it requires that the order of elimination of these be indifferent. This however is by no means always the case ($\varphi!\hat{u} = \hat{u}[\varphi!u]$, e.g., is a counter-example). In *Principia* such eliminations are always carried out by substitutions in the theorems corresponding to the definitions, so that it is chiefly the rule of substitution which would have to be proved.

I do not want, however, to go into any more details about either the formalism or the mathematical content of *Principia,*[2] but want to devote the subsequent portion of this essay to Russell's work concerning the analysis of the concepts and axioms underlying Mathematical Logic. In this field Russell has produced a great number of interesting ideas some of which are presented most clearly (or are contained only) in his earlier writings. I shall therefore frequently refer also to these earlier writings,

[2]Cf. in this respect Quine 1941.

although their content may partly disagree with Russell's present standpoint.

What strikes one as surprising in this field is Russell's pronouncedly realistic attitude, which manifests itself in many passages of his writings. "Logic is concerned with the real world just as truly as zoology, though with its more abstract and general features," he says, e.g., in his *Introduction to Mathematical Philosophy* (edition of 1920, p. 169). It is true, however, that this attitude has been gradually decreasing in the course of time[3] and also that it always was stronger in theory than in practice. When he started on a concrete problem, the objects to be analyzed, (e.g., the classes or propositions) soon for the most part turned into "logical fictions." Though perhaps this need not necessarily mean [according to the sense in which Russell uses this term] that these things do not exist, but only that we have no direct perception of them.

The analogy between mathematics and a natural science is enlarged upon by Russell also in another respect (in one of his earlier writings). He compares the axioms of logic and mathematics with the laws of nature and logical evidence with sense perception, so that the axioms need not necessarily be evident in themselves, but rather their justification lies (exactly as in physics) in the fact that they make it possible for these "sense perceptions" to be deduced; which of course would not exclude that they also have a kind of intrinsic plausibility similar to that in physics. I think that (provided "evidence" is understood in a sufficiently strict sense) this view has been largely justified by subsequent developments, and it is to be expected that it will be still more so in the future. It has turned out that (under the assumption that modern mathematics is consistent) the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception. Furthermore it seems likely that for deciding certain questions of abstract set theory and even for certain related questions of the theory of real numbers new axioms based on some hitherto unknown idea will be necessary. Perhaps also the apparently unsurmountable difficulties which some other mathematical problems have been presenting for many years are due to the fact that the necessary axioms have not yet been found. Of course, under these circumstances mathematics may lose a good deal of its "absolute certainty;" but, under the influence of the modern criticism of the foundations, this has already happened to a large extent. There is some resemblance

[3]The above quoted passage was left out in the later editions of the *Introduction*.

between this conception of Russell and Hilbert's "supplementing the data of mathematical intuition" by such axioms as, e.g., the law of excluded middle which are not given by intuition according to Hilbert's view; the borderline however between data and assumptions would seem to lie in different places according to whether we follow Hilbert or Russell.

An interesting example of Russell's analysis of the fundamental logical concepts is his treatment of the definite article "the". The problem is: what do the so-called descriptive phrases (i.e., phrases as, e.g., "the author of *Waverley*" or "the king of England") denote or signify[4] and what is the meaning of sentences in which they occur? The apparently obvious answer that, e.g., "the author of *Waverley*" signifies Walter Scott, leads to unexpected difficulties. For, if we admit the further apparently obvious axiom, that the signification of a composite expression, containing constituents which have themselves a signification, depends only on the signification of these constituents (not on the manner in which this signification is expressed), then it follows that the sentence "Scott is the author of *Waverley*" signifies the same thing as "Scott is Scott"; and this again leads almost inevitably to the conclusion that all true sentences have the same signification (as well as all false ones).[5] Frege actually drew this conclusion; and he meant it in an almost metaphysical sense, reminding one somewhat of the Eleatic doctrine of the "One." "The True" – according to Frege's view – is analyzed by us in different ways in different propositions; "the True" being the name he uses for the common signification of all true propositions (cf. 1892b: 35).

Now according to Russell, what corresponds to sentences in the outer world is facts. However, he avoids the term "signify" or "denote" and uses "indicate" instead (in his earlier papers he uses "express" or "being a symbol for"), because he holds that the relation between a sentence and a fact is quite different from that of a name to the thing named. Furthermore, he uses "denote" (instead of "signify") for the relation between things and names, so that "denote" and "indicate" together would correspond to Frege's "*bedeuten*". So, according to Russell's

[4]I use the term "signify" in the sequel because it corresponds to the German word "*bedeuten*" which Frege, who first treated the question under consideration, used in this connection.

[5]The only further assumptions one would need in order to obtain a rigorous proof would be: (1) that "$\varphi(a)$" and the proposition "a is the object which has the property φ and is identical with a" mean the same thing and (2) that every proposition "speaks about something," i.e., can be brought to the form $\varphi(a)$. Furthermore one would have to use the fact that for any two objects a, b, there exists a true proposition of the form $\varphi(a, b)$ as, e.g., $a \neq b$ or $a = a \cdot b = b$.

terminology and view, true sentences "indicate" facts and, correspondingly, false ones indicate nothing.[6] Hence Frege's theory would in a sense apply to false sentences, since they all indicate the same thing, namely nothing. But different true sentences may indicate many different things. Therefore this view concerning sentences makes it necessary either to drop the above-mentioned principle about the signification (i.e., in Russell's terminology the corresponding one about the denotation and indication) of composite expressions or to deny that a descriptive phrase denotes the object described. Russell did the latter[7] by taking the viewpoint that a descriptive phrase denotes nothing at all but has meaning only in context; for example, the sentence "the author of *Waverley* is Scotch", is defined to mean: "There exists exactly one entity who wrote *Waverley* and whoever wrote *Waverley* is Scotch." This means that a sentence involving the phrase "the author of *Waverley*" does not (strictly speaking) assert anything about Scott (since it contains no constituent denoting Scott), but is only a roundabout way of asserting something about the concepts occurring in the descriptive phrase. Russell adduces chiefly two arguments in favor of this view, namely (1) that a descriptive phrase may be meaningfully employed even if the object described does not exist (e.g., in the sentence: "The present king of France does not exist"). (2) That one may very well understand a sentence containing a descriptive phrase without being acquainted with the object described; whereas it seems impossible to understand a sentence without being acquainted with the objects about which something is being asserted. The fact that Russell does not consider this whole question of the interpretation of descriptions as a matter of mere linguistic conventions, but rather as a question of right and wrong, is another example of his realistic attitude, unless perhaps he was aiming at a merely psychological investigation of the actual processes of thought. As to the question in the logical sense, I cannot help feeling that the problem raised by Frege's puzzling conclusion has only been evaded by Russell's theory of descriptions and that there is something behind it which is not yet completely understood.

There seems to be one purely formal respect in which one may give preference to Russell's theory of descriptions. By defining the meaning

[6]From the indication (*Bedeutung*) of a sentence is to be distinguished what Frege called its meaning (*Sinn*) which is the conceptual correlate of the objectively existing fact (or "the True"). This one should expect to be in Russell's theory a possible fact (or rather the possibility of a fact), which would exist also in the case of a false proposition. But Russell, as he says, could never believe that such "curious shadowy" things really exist. Thirdly, there is also the psychological correlate of the fact which is called "signification" and understood to be the corresponding belief in Russell's latest book. "Sentence" in contradistinction to "proposition" is used to denote the mere combination of symbols.

[7]He made no explicit statement about the former; but it seems it would hold for the logical system of *Principia,* though perhaps more or less vacuously.

of sentences involving descriptions in the above manner, he avoids in his logical system any axioms about the particle "the", i.e., the analyticity of the theorems about "the" is made explicit; they can be shown to follow from the explicit definition of the meaning of sentences involving "the". Frege, on the contrary, has to assume an axiom about "the", which of course is also analytic, but only in the implicit sense that it follows from the meaning of the undefined terms. Closer examination, however, shows that this advantage of Russell's theory over Frege's subsists only as long as one interprets definitions as mere typographical abbreviations, not as introducing names for objects described by the definitions, a feature which is common to Frege and Russell.

I pass now to the most important of Russell's investigations in the field of the analysis of the concepts of formal logic, namely those concerning the logical paradoxes and their solution. By analyzing the paradoxes to which Cantor's set theory had led, he freed them from all mathematical technicalities, thus bringing to light the amazing fact that our logical intuitions (i.e., intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory. He then investigated where and how these common-sense assumptions of logic are to be corrected and came to the conclusion that the erroneous axiom consists in assuming that for every propositional function there exists the class of objects satisfying it, or that every propositional function exists "as a separate entity;"[8] by which is meant something separable from the argument (the idea being that propositional functions are abstracted from propositions which are primarily given) and also something distinct from the combination of symbols expressing the propositional function; it is then what one may call the notion or concept defined by it.[9] The existence of this concept already suffices for the paradoxes in their "intensional" form, where the concept of "not applying to itself" takes the place of Russell's paradoxical class.

Rejecting the existence of a class or concept in general, it remains to determine under what further hypotheses (concerning the propositional function) these entities do exist. Russell pointed out (1907: 29) two possible directions in which one may look for such a criterion, which he

[8]In Russell 1907: 29. If one wants to bring such paradoxes as "the liar" under this viewpoint, universal (and existential) propositions must be considered to involve the class of objects to which they refer.

[9]"Propositional function" (without the clause "as a separate entity") may be understood to mean a proposition in which one or several constituents are designated as arguments. One might think that the pair consisting of the proposition and the argument could then for all purposes play the rôle of the "propositional function as a separate entity," but it is to be noted that this pair (as one entity) is again a set or a concept and therefore need not exist.

called the zig-zag theory and the theory of limitation of size, respectively, and which might perhaps more significantly be called the intensional and the extensional theory. The second one would make the existence of a class or concept depend on the extension of the propositional function (requiring that it be not too big), the first one on its content or meaning (requiring a certain kind of "simplicity," the precise formulation of which would be the problem).

The most characteristic feature of the second (as opposed to the first) would consist in the non-existence of the universal class or (in the intensional interpretation) of the notion of "something" in an unrestricted sense. Axiomatic set theory as later developed by Zermelo and others can be considered as an elaboration of this idea as far as classes are concerned.[10] In particular the phrase "not too big" can be specified (as was shown by J. v. Neumann 1929: 227) to mean: not equivalent with the universe of all things, or, to be more exact, a propositional function can be assumed to determine a class when and only when there exists no relation (in intension, i.e., a propositional function with two variables) which associates in a one-to-one manner with each object, an object satisfying the propositional function and vice versa. This criterion, however, does not appear as the basis of the theory but as a consequence of the axioms and inversely can replace two of the axioms (the axiom of replacement and that of choice).

For the second of Russell's suggestions too, i.e., for the zig-zag theory, there has recently been set up a logical system which shares some essential features with this scheme, namely Quine's system (cf. 1937: 70). It is, moreover, not unlikely that there are other interesting possibilities along these lines.

Russell's own subsequent work concerning the solution of the paradoxes did not go in either of the two afore-mentioned directions pointed out by himself, but was largely based on a more radical idea, the "no-class theory," according to which classes or concepts *never* exist as real objects, and sentences containing these terms are meaningful only to such an extent as they can be interpreted as a *façon de parler*, a manner of speaking about other things (cf. p. [460]). Since in *Principia* and elsewhere, however, he formulated certain principles discovered in the course of the development of this theory as general logical principles without mentioning any longer their dependence on the no-class theory, I am going to treat of these principles first.

I mean in particular the vicious circle principle, which forbids a certain

[10]The intensional paradoxes can be dealt with, e.g., by the theory of simple types or the ramified hierarchy, which do not involve any undesirable restrictions if applied to concepts only and not to sets.

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kind of "circularity" which is made responsible for the paradoxes. The fallacy in these, so it is contended, consists in the circumstance that one defines (or tacitly assumes) totalities, whose existence would entail the existence of certain new elements of the same totality, namely elements definable only in terms of the whole totality. This led to the formulation of a principle which says that no totality can contain members definable only in terms of this totality, or members involving or presupposing this totality [vicious circle principle]. In order to make this principle applicable to the intensional paradoxes, still another principle had to be assumed, namely that "every propositional function presupposes the totality of its values" and therefore evidently also the totality of its possible arguments (cf. Whitehead and Russell 1910–13, 2: 39). [Otherwise the concept of "not applying to itself" would presuppose no totality (since it involves no quantifications),[11] and the vicious circle principle would not prevent its application to itself.] A corresponding vicious circle principle for propositional functions which says that nothing defined in terms of a propositional function can be a possible argument of this function is then a consequence (cf. Whitehead and Russell 1910–13, 1: 47, section 4). The logical system to which one is led on the basis of these principles is the theory of orders in the form adopted, e.g., in the first edition of *Principia,* according to which a propositional function which either contains quantifications referring to propositional functions of order n or can be meaningfully asserted of propositional functions of order n is at least of order $n+1$, and the range of significance of a propositional function as well as the range of a quantifier must always be confined to a definite order.

In the second edition of *Principia,* however, it is stated in the Introduction (pp. XI and XII) that "in a limited sense" also functions of a higher order than the predicate itself (therefore also functions defined in terms of the predicate as, e.g., in $p`\kappa \in \kappa$) can appear as arguments of a predicate of functions; and in appendix B such things occur constantly. This means that the vicious circle principle for propositional functions is virtually dropped. This change is connected with the new axiom that functions can occur in propositions only "through their values," i.e., extensionally, which has the consequence that any propositional function can take as an argument any function of appropriate type, whose extension is defined (no matter what order of quantifiers is used in the definition of this extension). There is no doubt that these things are quite unobjection-

[11]Quantifiers are the two symbols $(\exists x)$ and (x) meaning respectively, "there exists an object x" and "for all objects x." The totality of objects x to which they refer is called their range.

able even from the constructive standpoint (see below and p. [456]), provided that quantifiers are always restricted to definite orders. The paradoxes are avoided by the theory of simple types,[12] which in *Principia* is combined with the theory of orders (giving as a result the "ramified hierarchy") but is entirely independent of it and has nothing to do with the vicious circle principle (cf. pp. [464–5]).

Now as to the vicious circle principle proper, as formulated on p. [454], it is first to be remarked that, corresponding to the phrases "definable only in terms of," "involving," and "presupposing," we have really three different principles, the second and third being much more plausible than the first. It is the first form which is of particular interest, because only this one makes impredicative definitions[13] impossible and thereby destroys the derivation of mathematics from logic, effected by Dedekind and Frege, and a good deal of modern mathematics itself. It is demonstrable that the formalism of classical mathematics does not satisfy the vicious circle principle in its first form, since the axioms imply the existence of real numbers definable in this formalism only by reference to all real numbers. Since classical mathematics can be built up on the basis of *Principia* (including the axiom of reducibility), it follows that even *Principia* (in the first edition) does not satisfy the vicious circle principle in the first form, if "definable" means "definable within the system" and no methods of defining outside the system (or outside other systems of classical mathematics) are known except such as involve still more comprehensive totalities than those occurring in the systems.

I would consider this rather as a proof that the vicious circle principle is false than that classical mathematics is false, and this is indeed plausible also on its own account. For, first of all one may, on good grounds, deny that reference to a totality necessarily implies reference to all single elements of it or, in other words, that "all" means the same as an infinite logical conjunction. One may, e.g., follow Langford's (1927: 599) and Carnap's (1931: 103 [51 in this volume], and 1937: 162) suggestion to

[12]By the theory of simple types I mean the doctrine which says that the objects of thought (or, in another interpretation, the symbolic expressions) are divided into types, namely: individuals, properties of individuals, relations between individuals, properties of such relations, etc. (with a similar hierarchy for extensions), and that sentences of the form: "a has the property φ," "b bears the Relation R to c," etc. are meaningless, if a, b, c, R, φ are not of types fitting together. Mixed types (such as the class of all classes of finite types) are excluded. That the theory of simple types suffices for avoiding also the epistemological paradoxes is shown by a closer analysis of these. (Cf. Ramsey 1926a and Tarski 1935b: 399.)

[13]These are definitions of an object α by reference to a totality to which α itself (and perhaps also things definable only in terms of α) belong. As, e.g., if one defines a class α as the intersection of all classes satisfying a certain condition φ and then concludes that α is a subset also of such classes u as are defined in terms of α (provided they satisfy φ).

interpret "all" as meaning analyticity or necessity or demonstrability. There are difficulties in this view; but there is no doubt that in this way the circularity of impredicative definitions disappears.

Secondly, however, even if "all" means an infinite conjunction, it seems that the vicious circle principle in its first form applies only if the entities involved are constructed by ourselves. In this case there must clearly exist a definition (namely the description of the construction) which does not refer to a totality to which the object defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e., uniquely characterized)[14] only by reference to this totality (cf. Ramsey 1926a: 338 or 1931: 1). Such a state of affairs would not even contradict the second form of the vicious circle principle, since one cannot say that an object described by reference to a totality "involves" this totality, although the description itself does; nor would it contradict the third form, if "presuppose" means "presuppose for the existence" not "for the knowability."

So it seems that the vicious circle principle in its first form applies only if one takes the constructivistic (or nominalistic) standpoint[15] toward the objects of logic and mathematics, in particular toward propositions, classes and notions, e.g., if one understands by a notion a symbol together with a rule for translating sentences containing the symbol into such sentences as do not contain it, so that a separate object denoted by the symbol appears as a mere fiction.[16]

Classes and concepts may, however, also be conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions.

It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a

[14]An object a is said to be described by a propositional function $\varphi(x)$ is $\varphi(x)$ is true for $x=a$ and for no other object.

[15]I shall use in the sequel "constructivism" as a general term comprising both these standpoints and also such tendencies as are embodied in Russell's "no class" theory.

[16]One might think that this conception of notions is impossible, because the sentences into which one translates must also contain notions so that one would get into an infinite regress. This, however, does not preclude the possibility of maintaining the above viewpoint for all the more abstract notions, such as those of the second and higher types, or in fact for all notions except the primitive terms which might be only a very few.

satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the "data," i.e., in the latter case the actually occurring sense perceptions. Russell himself concludes in the last chapter of his book on *Meaning and Truth* (1940), though "with hesitation," that there exist "universals," but apparently he wants to confine this statement to concepts of sense perceptions, which does not help the logician. I shall use the term "concept" in the sequel exclusively in this objective sense. One formal difference between the two conceptions of notions would be that any two different definitions of the form $\alpha(x) = \varphi(x)$ can be assumed to define two different notions α in the constructivistic sense. (In particular this would be the case for the nominalistic interpretation of the term "notion" suggested above, since two such definitions give different rules of translation for propositions containing α.) For concepts, on the contrary, this is by no means the case, since the same thing may be described in different ways. It might even be that the axiom of extensionality[17] or at least something near to it holds for concepts. The difference may be illustrated by the following definition of the number two: "Two is the notion under which fall all pairs and nothing else." There is certainly more than one notion in the constructivistic sense satisfying this condition, but there might be one common "form" or "nature" of all pairs.

Since the vicious circle principle, in its first form does apply to constructed entities, impredicative definitions and the totality of all notions or classes or propositions are inadmissible in constructivistic logic. What an impredicative definition would require is to construct a notion by a combination of a set of notions to which the notion to be formed itself belongs. Hence if one tries to effect a retranslation of a sentence containing a symbol for such an impredicatively defined notion it turns out that what one obtains will again contain a symbol for the notion in question (cf. Carnap 1931: 103 [51 in this volume] and 1937: 162). At least this is so if "all" means an infinite conjunction; but Carnap's and Langford's idea (mentioned on pp. [455–6]) would not help in this connection, because "demonstrability," if introduced in a manner compatible with the constructivistic standpoint towards notions, would have to be split into a hierarchy of orders, which would prevent one from obtaining the desired results.[18] As Chwistek (1933: 367) has shown, it is even possible under

[17] I.e., that no two different properties belong to exactly the same things, which, in a sense, is a counterpart to Leibniz's *Principium identitatis indiscernibilium,* which says no two different things have exactly the same properties.

[18] Nevertheless the scheme is interesting because it again shows the constructibility of notions which can be meaningfully asserted of notions of arbitrarily high order.

certain assumptions admissible within constructivistic logic to derive an actual contradiction from the unrestricted admission of impredicative definitions. To be more specific, he has shown that the system of simple types becomes contradictory if one adds the "axiom of intensionality" which says (roughly speaking) that to different definitions belong different notions. This axiom, however, as has just been pointed out, can be assumed to hold for notions in the constructivistic sense.

Speaking of concepts, the aspect of the question is changed completely. Since concepts are supposed to exist objectively, there seems to be objection neither to speaking of all of them (cf. p. [461]) nor to describing some of them by reference to all (or at least all of a given type). But, one may ask, isn't this view refutable also for concepts because it leads to the "absurdity" that there will exist properties φ such that $\varphi(a)$ consists in a certain state of affairs involving all properties (including φ itself and properties defined in terms of φ), which would mean that the vicious circle principle does not hold even in its second form for concepts or propositions? There is no doubt that the totality of all properties (or of all those of a given type) does lead to situations of this kind, but I don't think they contain any absurdity.[19] It is true that such properties φ [or such propositions $\varphi(a)$] will have to contain themselves as constituents of their content [or of their meaning], and in fact in many ways, because of the properties defined in terms of φ; but this only makes it impossible to construct their meaning (i.e., explain it as an assertion about sense perceptions or any other non-conceptual entities), which is no objection for one who takes the realistic standpoint. Nor is it self-contradictory that a proper part should be identical (not merely equal) to the whole, as is seen in the case of structures in the abstract sense. The structure of the series of integers, e.g., contains itself as a proper part and it is easily seen that there exist also structures containing infinitely many different parts, each containing the whole structure as a part. In addition there exist, even within the domain of constructivistic logic, certain approximations to this self-reflexivity of impredicative properties, namely propositions which contain as parts of their meaning not themselves but their own formal demonstrability (cf. Gödel 1931: 173 or Carnap 1937, §35). Now formal demonstrability of a proposition (in case the axioms and rules of inference are correct) implies this proposition

[19]The formal system corresponding to this view would have, instead of the axiom of reducibility, the rule of substitution for functions described, e.g., in Hilbert-Bernays 1934–9, 1: 90, applied to variables of any type, together with certain axioms of intensionality required by the concept of property which, however, would be weaker than Chwistek's. It should be noted that this view does not necessarily imply the existence of concepts which cannot be expressed in the system, if combined with a solution of the paradoxes along the lines indicated on p. [466].

and in many cases is equivalent to it. Furthermore, there doubtlessly exist sentences referring to a totality of sentences to which they themselves belong as, e.g., the sentence: "Every sentence (of a given language) contains at least one relation word."

Of course this view concerning the impredicative properties makes it necessary to look for another solution of the paradoxes, according to which the fallacy (i.e., the underlying erroneous axiom) does not consist in the assumption of certain self-reflexivities of the primitive terms but in other assumptions about these. Such a solution may be found for the present in the simple theory of types and in the future perhaps in the development of the ideas sketched on pp. [452–3 and 466]. Of course, all this refers only to concepts. As to notions in the constructivistic sense there is no doubt that the paradoxes are due to a vicious circle. It is not surprising that the paradoxes should have different solutions for different interpretations of the terms occurring.

As to classes in the sense of pluralities or totalities it would seem that they are likewise not created but merely described by their definitions and that therefore the vicious circle principle in the first form does not apply. I even think there exist interpretations of the term "class" (namely as a certain kind of structures), where it does not apply in the second form either.[20] But for the development of all contemporary mathematics one may even assume that it does apply in the second form, which for classes as mere pluralities is, indeed, a very plausible assumption. One is then led to something like Zermelo's axiom system for set theory, i.e., the sets are split up into "levels" in such a manner that only sets of lower levels can be elements of sets of higher levels (i.e., $x \in y$ is always false if x belongs to a higher level than y). There is no reason for classes in this sense to exclude mixtures of levels in one set and transfinite levels. The place of the axiom of reducibility is now taken by the axiom of classes [Zermelo's *Aussonderungsaxiom*] which says that for each level there exists for an arbitrary propositional function $\varphi(x)$ the set of those x of this level for which $\varphi(x)$ is true, and this seems to be implied by the concept of classes as pluralities.

Russell adduces two reasons against the extensional view of classes, namely the existence of (1) the null class, which cannot very well be a collection, and (2) the unit classes, which would have to be identical with their single elements. But it seems to me that these arguments could, if anything, at most prove that the null class and the unit classes (as distinct from their only element) are fictions (introduced to simplify the calculus like the points at infinity in geometry), not that all classes are fictions.

[20]Ideas tending in this direction are contained in Mirimanoff 1917a: 37–52, 1917b: 209–17, 1920: 29–52.

But in Russell the paradoxes had produced a pronounced tendency to build up logic as far as possible without the assumption of the objective existence of such entities as classes and concepts. This led to the formulation of the aforementioned "no class theory," according to which classes and concepts were to be introduced as a *façon de parler.* But propositions, too, (in particular those involving quantifications; Russell 1906a: 627) were later on largely included in this scheme, which is but a logical consequence of this standpoint, since, e.g., universal propositions as objectively existing entities evidently belong to the same category of idealistic objects as classes and concepts and lead to the same kind of paradoxes, if admitted without restrictions. As regards classes this program was actually carried out, i.e., the rules for translating sentences containing class names or the term "class" into such as do not contain them were stated explicitly; and the basis of the theory, i.e., the domain of sentences into which one has to translate is clear, so that classes can be dispensed with (within the system *Principia*), but only if one assumes the existence of a concept whenever one wants to construct a class. When it comes to concepts and the interpretation of sentences containing this or some synonymous term, the state of affairs is by no means as clear. First of all, some of them (the primitive predicates and relations such as "red" or "colder") must apparently be considered as real objects;[21] the rest of them (in particular according to the second edition of *Principia,* all notions of a type higher than the first and therewith all logically interesting ones) appear as something constructed (i.e., as something not belonging to the "inventory" of the world); but neither the basic domain of propositions in terms of which finally everything is to be interpreted, nor the method of interpretation is as clear as in the case of classes (see below).

This whole scheme of the no-class theory is of great interest as one of the few examples, carried out in detail, of the tendency to eliminate assumptions about the existence of objects outside the "data" and to replace them by constructions on the basis of these data.[22] The result has been in this case essentially negative; i.e., the classes and concepts introduced in this way do not have all the properties required for their use in mathematics, unless one either introduces special axioms about the data (e.g., the axiom of reducibility), which in essence already mean the existence in the data of the kind of objects to be constructed, or makes the fiction that one can form propositions of infinite (and even non-

[21] In Appendix C of *Principia* a way is sketched by which these also could be constructed by means of certain similarity relations between atomic propositions, so that these latter would be the only ones remaining as real objects.

[22] The "data" are to be understood in a relative sense here, i.e., in our case as logic without the assumption of the existence of classes and concepts.

denumerable) length (cf. Ramsey 1926a: 338 or 1931: 1), i.e., operates with truth-functions of infinitely many arguments, regardless of whether or not one can construct them. But what else is such an infinite truth-function but a special kind of an infinite extension (or structure) and even a more complicated one than a class, endowed in addition with a hypothetical meaning, which can be understood only by an infinite mind? All this is only a verification of the view defended above that logic and mathematics (just as physics) are built up on axioms with a real content which cannot be "explained away."

What one can obtain on the basis of the constructivistic attitude is the theory of orders (cf. p. [454]); only now (and this is the strong point of the theory) the restrictions involved do not appear as *ad hoc* hypotheses for avoiding the paradoxes, but as unavoidable consequences of the thesis that classes, concepts, and quantified propositions do not exist as real objects. It is not as if the universe of things were divided into orders and then one were prohibited to speak of all orders; but, on the contrary, it is possible to speak of all existing things; only, classes and concepts are not among them; and if they are introduced as a *façon de parler,* it turns out that this very extension of the symbolism gives rise to the possibility of introducing them in a more comprehensive way, and so on indefinitely. In order to carry out this scheme one must, however, presuppose arithmetic (or something equivalent) which only proves that not even this restricted logic can be built up on nothing.

In the first edition of *Principia,* where it was a question of actually building up logic and mathematics, the constructivistic attitude was, for the most part, abandoned, since the axiom of reducibility for types higher than the first together with the axiom of infinity makes it absolutely necessary that there exist primitive predicates of arbitrarily high types. What is left of the constructive attitude is only: (1) The introduction of classes as a *façon de parler;* (2) the definition of $\sim$, $\vee$, $\cdot$, etc., as applied to propositions containing quantifiers (which incidentally proved its fecundity in a consistency proof for arithmetic); (3) the step-by-step construction of functions of orders higher than 1, which, however, is superfluous owing to the axiom of reducibility; (4) the interpretation of definitions as mere typographical abbreviations, which makes every symbol introduced by definition an incomplete symbol (not one naming an object described by the definition). But the last item is largely an illusion, because, owing to the axiom of reducibility, there always exist real objects in the form of primitive predicates, or combinations of such, corresponding to each defined symbol. Finally also Russell's theory of descriptions is something belonging to the constructivistic order of ideas.

In the second edition of *Principia* (or to be more exact, in the introduc-

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tion to it) the constructivistic attitude is resumed again. The axiom of reducibility is dropped and it is stated explicitly that all primitive predicates belong to the lowest type and that the only purpose of variables (and evidently also of constants) of higher orders and types is to make it possible to assert more complicated truth-functions of atomic propositions,[23] which is only another way of saying that the higher types and orders are solely a *façon de parler.* This statement at the same time informs us of what kind of propositions the basis of the theory is to consist, namely of truth-functions of atomic propositions.

This, however, is without difficulty only if the number of individuals and primitive predicates is finite. For the opposite case (which is chiefly of interest for the purpose of deriving mathematics), Ramsey (cf. Ramsey 1926a: 338 or 1931: 1) took the course of considering our inability to form propositions of infinite length as a "mere accident," to be neglected by the logician. This of course solves (or rather cuts through) the difficulties; but it is to be noted that, if one disregards the difference between finite and infinite in this respect, there exists a simpler and at the same time more far-reaching interpretation of set theory (and therewith of mathematics). Namely, in case of a finite number of individuals, Russell's *aperçu* that propositions about classes can be interpreted as propositions about their elements becomes literally true, since, e.g., "$x \in m$" is equivalent to "$x = a_1 \vee x = a_2 \vee \ldots \vee x = a_k$" where the a_i are the elements of m; and "there exists a class such that . . ." is equivalent to "there exist individuals $x_1, x_2, \ldots, x_n$ such that . . . ,"[24] provided n is the number of individuals in the world and provided we neglect for the moment the null class which would have to be taken care of by an additional clause. Of course, by an iteration of this procedure one can obtain classes of classes, etc., so that the logical system obtained would resemble the theory of simple types except for the circumstance that mixture of types would be possible. Axiomatic set theory appears, then, as an extrapolation of this scheme for the case of infinitely many individuals or an infinite iteration of the process of forming sets.

Ramsey's viewpoint is, of course, everything but constructivistic, unless one means constructions of an infinite mind. Russell, in the second edition of *Principia,* took a less metaphysical course by confining himself to such truth-functions as can actually be constructed. In this way one is again led to the theory of orders, which, however, appears now in a new light, namely as a method of constructing more and more complicated

[23] I.e., propositions of the form $S(a), R(a, b)$, etc., where S, R are primitive predicates and a, b individuals.

[24] The x_i may, of course, as always, be partly or wholly identical with each other.

truth-functions of atomic propositions. But this procedure seems to presuppose arithmetic in some form or other (see next paragraph).

As to the question of how far mathematics can be built up on this basis (without any assumptions about the data – i.e., about the primitive predicates and individuals – except, as far as necessary, the axiom of infinity), it is clear that the theory of real numbers in its present form cannot be obtained.[25] As to the theory of integers, it is contended in the second edition of *Principia* that it can be obtained. The difficulty to be overcome is that in the definition of the integers as "those cardinals which belong to every class containing 0 and containing $x+1$ if containing x," the phrase "every class" must refer to a given order. So one obtains integers of different orders, and complete induction can be applied to integers of order n only for properties of order n; whereas it frequently happens that the notion of integer itself occurs in the property to which induction is applied. This notion, however, is of order $n+1$ for the integers of order n. Now, in Appendix B of the second edition of *Principia,* a proof is offered that the integers of any order higher than 5 are the same as those of order 5, which of course would settle all difficulties. The proof as it stands, however, is certainly not conclusive. In the proof of the main lemma *89.16, which says that every subset α (of arbitrary high order)[26] of an inductive class β of order 3 is itself an inductive class of order 3, induction is applied to a property of β involving α [namely $\alpha - \beta \neq \Lambda$, which, however, should read $\alpha - \beta \sim\in \text{Induct}_2$ because (3) is evidently false]. This property, however, is of an order >3 if α is of an order >3. So the question whether (or to what extent) the theory of integers can be obtained on the basis of the ramified hierarchy must be considered as unsolved at the present time. It is to be noted, however, that, even in case this question should have a positive answer, this would be of no value for the problem whether arithmetic follows from logic, if propositional functions of order n are defined (as in the second edition of *Principia*) to be certain finite (though arbitrarily complex) combinations (of quantifiers, propositional connectives, etc.), because then the notion of finiteness has to be presupposed, which fact is concealed only by taking such complicated notions as "propositional function of order n" in an unanalyzed form as primitive terms of the formalism and giving their definition only in ordinary language. The reply may perhaps be

[25] As to the question how far it is possible to build up the theory of real numbers, presupposing the integers, cf. Weyl 1918.

[26] That the variable α is intended to be of undetermined order is seen from the later applications of *89.17 and from the note to *89.17. The main application is in line (2) of the proof of *89.24, where the lemma under consideration is needed for α's of arbitrarily high orders.

offered that in *Principia* the notion of a propositional function of order n is neither taken as primitive nor defined in terms of the notion of a finite combination, but rather quantifiers referring to propositional functions of order n (which is all one needs) are defined as certain infinite conjunctions and disjunctions. But then one must ask: Why doesn't one define the integers by the infinite disjunction: $x=0 \vee x=0+1 \vee x=0+1+1 \vee \ldots$ *ad infinitum,* saving in this way all the trouble connected with the notion of inductiveness? This whole objection would not apply if one understands by a propositional function of order n one "obtainable from such truth-functions of atomic propositions as presuppose for their definition no totalities except those of the propositional functions of order $<n$ and of individuals"; this notion, however, is somewhat lacking in precision.

The theory of orders proves more fruitful if considered from a purely mathematical standpoint, independently of the philosophical question whether impredicative definitions are admissible. Viewed in this manner, i.e., as a theory built up within the framework of ordinary mathematics, where impredicative definitions are admitted, there is no objection to extending it to arbitrarily high transfinite orders. Even if one rejects impredicative definitions, there would, I think, be no objection to extend it to such transfinite ordinals as can be constructed within the framework of finite orders. The theory in itself seems to demand such an extension since it leads automatically to the consideration of functions in whose definition one refers to all functions of finite orders, and these would be functions of order ω. Admitting transfinite orders, an axiom of reducibility can be proved. This, however, offers no help to the original purpose of the theory, because the ordinal α – such that every propositional function is extensionally equivalent to a function of order α – is so great, that it presupposes impredicative totalities. Nevertheless, so much can be accomplished in this way, that all impredicativities are reduced to one special kind, namely the existence of certain large ordinal numbers (or, well-ordered sets) and the validity of recursive reasoning for them. In particular, the existence of a well-ordered set, of order type ω_1 already suffices for the theory of real numbers. In addition this transfinite theorem of reducibility permits the proof of the consistency of the Axiom of Choice, of Cantor's Continuum-Hypothesis and even of the generalized Continuum-Hypothesis (which says that there exists no cardinal number between the power of any arbitrary set and the power of the set of its subsets) with the axioms of set theory as well as of *Principia.*

I now come in somewhat more detail to the theory of simple types which appears in *Principia* as combined with the theory of orders; the former is, however, (as remarked above) quite independent of the latter, since mixed types evidently do not contradict the vicious circle principle in any way. Accordingly, Russell also based the theory of simple types on entirely different reasons. The reason adduced (in addition to its "consonance with common sense") is very similar to Frege's, who, in his system, already had assumed the theory of simple types for functions, but failed to avoid the paradoxes, because he operated with classes (or rather functions in extension) without any restriction. This reason is that (owing to the variable it contains) a propositional function is something ambiguous (or, as Frege says, something unsaturated, wanting supplementation) and therefore can occur in a meaningful proposition only in such a way that this ambiguity is eliminated (e.g., by substituting a constant for the variable or applying quantification to it). The consequences are that a function cannot replace an individual in a proposition, because the latter has no ambiguity to be removed, and that functions with different kinds of arguments (i.e., different ambiguities) cannot replace each other; which is the essence of the theory of simple types. Taking a more nominalistic viewpoint (such as suggested in the second edition of *Principia* and in *Meaning and Truth*) one would have to replace "proposition" by "sentence" in the foregoing considerations (with corresponding additional changes). But in both cases, this argument clearly belongs to the order of ideas of the "no class" theory, since it considers the notions (or propositional functions) as something constructed out of propositions or sentences by leaving one or several constituents of them undetermined. Propositional functions in this sense are so to speak "fragments" of propositions, which have no meaning in themselves, but only insofar as one can use them for forming propositions by combining several of them, which is possible only if they "fit together," i.e., if they are of appropriate types. But, it should be noted that the theory of simple types (in contradistinction to the vicious circle principle) cannot in a strict sense follow from the constructive standpoint, because one might construct notions and classes in another way, e.g., as indicated on p. [462], where mixtures of types are possible. If on the other hand one considers concepts as real objects, the theory of simple types is not very plausible, since what one would expect to be a concept (such as, e.g., "transitivity" or the number two) would seem to be something behind all its various "realizations" on the different levels and therefore does not exist according to the theory of types. Nevertheless, there seems to be some truth behind this idea of realizations of the same concept on various levels, and one might, therefore, expect the theory of simple types to prove useful or necessary at least as a stepping-stone for a more satisfactory system, a way in which it has already been used by Quine (cf. 1937: 70). Also

Russell's "typical ambiguity" is a step in this direction. Since, however, it only adds certain simplifying symbolic conventions to the theory of types, it does not *de facto* go beyond this theory.

It should be noted that the theory of types brings in a new idea for the solution of the paradoxes, especially suited to their intensional form. It consists in blaming the paradoxes not on the axiom that every propositional function defines a concept or class, but on the assumption that every concept gives a meaningful proposition, if asserted for any arbitrary object or objects as arguments. The obvious objection that every concept can be extended to all arguments, by defining another one which gives a false proposition whenever the original one was meaningless, can easily be dealt with by pointing out that the concept "meaningfully applicable" need not itself be always meaningfully applicable.

The theory of simple types (in its realistic interpretation) can be considered as a carrying through of this scheme, based, however, on the following additional assumption concerning meaningfulness: "Whenever an object x can replace another object y in one meaningful proposition, it can do so in every meaningful proposition."[27] This of course has the consequence that the objects are divided into mutually exclusive ranges of significance, each range consisting of those objects which can replace each other; and that therefore each concept is significant only for arguments belonging to one of these ranges, i.e., for an infinitely small portion of all objects. What makes the above principle particularly suspect, however, is that its very assumption makes its formulation as a meaningful proposition impossible,[28] because x and y must then be confined to definite ranges of significance which are either the same or different, and in both cases the statement does not express the principle or even part of it. Another consequence is that the fact that an object x is (or is not) of a given type also cannot be expressed by a meaningful proposition.

It is not impossible that the idea of limited ranges of significance could be carried out without the above restrictive principle. It might even turn out that it is possible to assume every concept to be significant everywhere except for certain "singular points" or "limiting points," so that the paradoxes would appear as something analogous to dividing by zero. Such a system would be most satisfactory in the following respect: our logical intuitions would then remain correct up to certain minor correc-

[27]Russell formulates a somewhat different principle with the same effect (Whitehead and Russell 1910–13, 1: 95).

[28]This objection does not apply to the symbolic interpretation of the theory of types, spoken of on p. [465], because there one does not have objects but only symbols of different types.

tions, i.e., they could then be considered to give an essentially correct, only somewhat "blurred," picture of the real state of affairs. Unfortunately the attempts made in this direction have failed so far;[29] on the other hand, the impossibility of this scheme has not been proved either, in spite of the strong inconsistency theorems of Kleene and Rosser (1935: 630).

In conclusion I want to say a few words about the question whether (and in which sense) the axioms of *Principia* can be considered to be analytic. As to this problem it is to be remarked that analyticity may be understood in two senses. First, it may have the purely formal sense that the terms occurring can be defined (either explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law. In this sense even the theory of integers is demonstrably non-analytic, provided that one requires of the rules of elimination that they allow one actually to carry out the elimination in a finite number of steps in each case.[30] Leaving out this condition by admitting, e.g., sentences of infinite (and non-denumerable) length as intermediate steps of the process of reduction, all axioms of *Principia* (including the axioms of choice, infinity and reducibility) could be proved to be analytic for certain interpretations (by considerations similar to those referred to on p. [462]).[31] But this observation is of doubtful value, because the whole of mathematics as applied to sentences of infinite length has to be presupposed in order to prove this analyticity, e.g., the axiom of choice can be proved to be analytic only if it is assumed to be true.

In a second sense a proposition is called analytic if it holds, "owing to the meaning of the concepts occurring in it," where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental).[32] It would seem that all axioms of *Principia,* in the first edition, (except the axiom of infinity) are in this sense analytic for certain interpretations of the primitive terms, namely if the term "predicative function" is replaced either by "class" (in the extensional sense) or (leaving out the axiom of choice) by "concept," since nothing can express better the

[29]A formal system along these lines is Church's "A Set of Postulates for the Foundation of Logic" (1932: 346; 1933: 839), where, however, the underlying idea is expressed by the somewhat misleading statement that the law of excluded middle is abandoned. However, this system has been proved to be inconsistent. See Kleene and Rosser 1935.

[30]Because this would imply the existence of a decision-procedure for all arithmetical propositions. Cf. Turing 1937: 230.

[31]Cf. also Ramsey (1926a: 338 or 1931: 1), where, however, the axiom of infinity cannot be obtained, because it is interpreted to refer to the individuals in the world.

[32]The two significations of the term *analytic* might perhaps be distinguished as tautological and analytic.

meaning of the term "class" than the axiom of the classes (cf. p. [459]) and the axiom of choice, and since, on the other hand, the meaning of the term "concept" seems to imply that every propositional function defines a concept.[33] The difficulty is only that we don't perceive the concepts of "concept" and of "class" with sufficient distinctness, as is shown by the paradoxes. In view of this situation, Russell took the course of considering both classes and concepts (except the logically uninteresting primitive predicates) as non-existent and of replacing them by constructions of our own. It cannot be denied that this procedure has led to interesting ideas and to results valuable also for one taking the opposite viewpoint. On the whole, however, the outcome has been that only fragments of Mathematical Logic remain, unless the things condemned are reintroduced in the form of infinite propositions or by such axioms as the axiom of reducibility which (in case of infinitely many individuals) is demonstrably false unless one assumes either the existence of classes or of infinitely many "*qualitates occultae.*" This seems to be an indication that one should take a more conservative course, such as would consist in trying to make the meaning of the terms "class" and "concept" clearer, and to set up a consistent theory of classes and concepts as objectively existing entities. This is the course which the actual development of Mathematical Logic has been taking and which Russell himself has been forced to enter upon in the more constructive parts of his work. Major among the attempts in this direction (some of which have been quoted in this essay) are the simple theory of types (which is the system of the first edition of *Principia* in an appropriate interpretation) and axiomatic set theory, both of which have been successful at least to this extent, that they permit the derivation of modern mathematics and at the same time avoid all known paradoxes. Many symptoms show only too clearly, however, that the primitive concepts need further elucidation.

It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that Mathematical Logic has up to now remained so far behind the high expectations of Peano and others who (in accordance with Leibniz's claims) had hoped

[33]This view does not contradict the opinion defended above that mathematics is based on axioms with a real content, because the very existence of the concept of, e.g., "class" constitutes already such an axiom; since, if one defined, e.g., "class" and "$\in$" to be "the concepts satisfying the axioms," one would be unable to prove their existence. "Concept" could perhaps be defined in terms of "proposition" (cf. p. [465]) (although I don't think that this would be a natural procedure); but then certain axioms about propositions, justifiable only with reference to the undefined meaning of this term, will have to be assumed. It is to be noted that this view about analyticity makes it again possible that every mathematical proposition could perhaps be reduced to a special case of $a = a$, namely if the reduction is effected not in virtue of the definitions of the terms occurring, but in virtue of their meaning, which can never be completely expressed in a set of formal rules.

that it would facilitate theoretical mathematics to the same extent as the decimal system of numbers has facilitated numerical computations. For how can one expect to solve mathematical problems systematically by mere analysis of the concepts occurring, if our analysis so far does not even suffice to set up the axioms? But there is no need to give up hope. Leibniz did not in his writings about the *Characteristica universalis* speak of a utopian project; if we are to believe his words he had developed this calculus of reasoning to a large extent, but was waiting with its publication till the seed could fall on fertile ground (1875–90, 7: 12; Vacca 1903: 72; Leibniz 1923–, 1: preface). He went even so far (1875–90, 7: 187) as to estimate the time which would be necessary for his calculus to be developed by a few select scientists to such an extent "that humanity would have a new kind of an instrument increasing the powers of reason far more than any optical instrument has ever aided the power of vision." The time he names is five years, and he claims that his method is not any more difficult to learn than the mathematics or philosophy of his time. Furthermore, he said repeatedly that, even in the rudimentary state to which he had developed the theory himself, it was responsible for all his mathematical discoveries; which, one should expect, even Poincaré would acknowledge as a sufficient proof of its fecundity.[34]

[34] I wish to express my thanks to Professor Alonzo Church, of Princeton University, who helped me find the correct English expressions in a number of places.

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