On Formalization

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VI.—ON FORMALIZATION

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1. *Systematization*

THE most striking results of formalization occur in logic and mathematics.

Here formalization provides at least one kind of systematization. We are led to believe that there is a fairly simple axiom system from which it is possible to derive almost all mathematical theorems and truths mechanically. This is at present merely a theoretical possibility, for no serious attempts seem to have been made to prove, for instance, all the theorems of an elementary textbook of calculus. Nevertheless, we seem to get a feeling of grandeur from the realization that a simple axiom system which we can quite easily memorize by heart embodies, in a sense, practically all the mathematical truths. It is not very hard to get to know the axiom system so well that people would say you understood the system. Unfortunately just to be able thus to understand the system neither gives you very deep insight into the nature of mathematics nor makes you a very good mathematician.

To say that physics uses the experimental method is not to say much about physics. To say that all theorems of mathematics can be proved from certain axioms by chains of syllogism (or *modus ponens*) is to say just as little about mathematics. Merely knowing the experimental method is not knowing the whole of physics; merely knowing an axiom system adequate for developing mathematics is not knowing the whole of mathematics.

There is another kind of systematization which is less superficial than learning the axiom system. It is an intuitive grasp of the whole field, a vivid picture of the whole structure in your mind such as a good chess player would have of the game of chess. This second kind of systematization is something that formalization (or at least formalization alone) would not provide us.

If we had never used logistic systems at all, the many interesting results about logistic systems (such as those of Skolem, Herbrand, and Gödel) would, of course, never have been expressed in the specific form in which they are now being expressed. But it is not certain that essentially the same results might not have been attained, though in other contexts and as the results about other things. Nevertheless, axiomatics or the axiomatic method has a strong appeal in that here we seem to be able to prove sweeping conclusions about whole fields. For many of us a significant theorem about a whole field appears more important than particular theorems in the field. In generating systems, formalization serves the function of enabling us to talk precisely about whole fields of learning.

2. *Communication*

It is hard to say whether in general formalization renders a theory or a proof easier to understand.

Consider, for example, an oral sketch of a newly discovered proof, an abstract designed to communicate just the basic idea of the proof, an article presenting the proof to people working on related problems, a textbook formulation of the same, and a presentation of it after the manner of *Principia Mathematica*. The proof gets more and more thoroughly formalized as we go from an earlier version to a later. It is, however, questionable whether in general a more completely formalized version is clearer or serves better as a means of communication. Each step of it should be easier to follow since it involves no jumps. But even this is not certain, for there are many jumps which we are so used to making that we find it more natural to make the jumps than not to. Or alternatively, we may say that the step actually does not involve jumps and that our formal proof suggests that it does only because our formal system is defective as a map of our intuitive logic.

Who finds which proof easier to follow or who understands which proof in a shorter while depends pretty much on what background the man happens to have. In general, the better acquainted one is with the problem, the easier he finds the use of a more sketchy proof. But there is also a certain limit beyond which even the expert in the matter can no longer supply for himself the missing details. Moreover, there is always the possibility that the presentation would be much shorter if it were not so short. It seems safe, however, to say that a more thoroughly formalized proof is generally longer, provided that we do not appeal to abbreviations in its presentation and the less formalized version does not waste words.

We are all familiar with requests to explain a physical theory without using mathematics, to convey the basic idea of a proof without using symbols. Therefore, it would seem that in general the plain words or the less technical language provide a more efficient means of communication. Actually, however, we can easily think of examples which would indicate that this is not quite true.

To put thoughts on physics into mathematical symbols is one way of formalization. Through accumulation and tradition this way of formalization has also become a powerful way of communication: for those who understand the language, a short formula may express more precisely thought which could only be explained by many pages of ordinary words, and much less satisfactorily. Sometimes it becomes practically impossible to avoid the mathematical language in communicating with others. An elderly English political figure complains that none of the many eminent physicists with whom he has corresponded is courageous enough to pass any definite judgment on his proposed new theory of ether. Then he stresses the similarity between his theory and the concluding paragraph of a recent article by Dirac, and proceeds to discard as non-essential the accompanying mathematical passages in Dirac's article. It may be presumed that if he had also included comparable non-essential mathematical passages in his theory, he would have received more definite responses.

3. *Clarity and Consolidation*

Does formalization help us to analyse and clarify concepts ?

Often in formalizing ordinary concepts, we appear to have platitudes restated in pedantic obscurity; for instance, the mathematical definition of the continuity of a curve or the technical definition of the notion of effective computability. Moreover, the exact formalizations almost always distort our ordinary language at one place or another. For example, it has been pointed out that Russell's theory of descriptions does not apply to sentences such as "the whale is a mammal", and that sometimes in ordinary use the sentence "the king of France is bald" is neither taken as true nor taken as false.

In scientific investigations, we often recognize the advantage and even necessity of paying the price of considerable deviation from ordinary use of words in order to reach fairly precise terminology and notation. But, in what sense is, for instance, the technical notion of effective computability clearer than the corresponding common sense concept ? Ordinarily, we would tend to say that the technical notion is *less* clear because it is more difficult to learn and a concept is clearer if and only if it is easier. We might speak of different kinds of clarity just as Mill speaks of different kinds of pleasure. Then we can also speak of a principle of preference: Only those who have experienced the feeling of clarity both of the ordinary notion and of the technical one are qualified to judge which is really clearer. And then, we hope, they will find the formalized notion clearer.

Perhaps we should also say that which definition of a term is clearer depends partly on the purposes we want the term to serve, and partly on our familiarity with the notions involved in each definition. The main advantage of the more articulate definition of a notion is, presumably, that it is sharper: for example, there are many cases where we can give a definite answer to the question whether certain given functions are effectively computable, only after we have made use of the technical notion of computability.

There are many cases where we could neither ask a univocal question nor obtain a univocal answer until we possessed the formalized notion. For example, we needed an exact definition of continuous curves before we could ask and answer the question whether there are space-filling continuous curves. And it was necessary first to formalize the notions of completeness and decidability before a negative answer could be given to the question whether number theory is complete or decidable.

Significant formalization of a concept involves analysis of the concept, not so much in the sense of analysis when we say that being a bachelor entails being unmarried, but more in the sense that an analysis of the problem of squaring the circle is provided by the proof of its unsolvability. When formalization is performed at such a level, it does serve to clarify and explicate concepts.

Another function of formalization is the clarification and consolidation of arguments or proofs. Sometimes we are not quite sure whether we have understood a certain given proof, sometimes we understand a proof once but fail to understand it again when reading it a few days later. Then there often comes the desire to work over the proof thoroughly, to make explicit all the implicit steps involved, and to write down the expanded result once and for all. With some people this desire to formalize and expand proofs may become a habit and a handicap to studying certain branches of mathematics. Yet occasional indulgence in this kind of thoroughness need not be a harmful thing.

In certain cases, there is no sharp line between formalizing and discovering a proof. There are many cases where essentially incomplete sketches, sometimes containing errors as well, get expanded and made into more exact proofs. Sometimes it is not until we have the thoroughly worked out proof on hand that we begin to perceive a connexion between it and the existing hint or sketch. Sometimes it seems hard to decide whether to consider the sketcher or the formalizer the true discoverer of the proof.

4. *Rigour*

In a sense, to formalize is to make rigorous.

There was Berkeley's attack on the mathematicians of his day entitled: "The analyst: or, a discourse addressed to an infidel mathematician. Wherein it is examined whether the object, principles, and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith." There is the long story of how Lagrange, Cauchy, Weierstrass, and others strove to formalize exactly the basic notions of limits, continuity, derivatives, etc., providing thereby rigorous (though not necessarily reliable) foundations for mathematical analysis.

In the contemporary scene, we have logicians deploring how carelessly ordinary mathematicians use their words and symbols. Some logicians are puzzled that so many apparent confusions in mathematics do not lead more often to serious errors. On the other hand, mathematicians in turn complain about the inaccuracy of alleged proofs of mathematical theorems by physicists and engineers.

In the other direction, physicists consider that mathematicians are wasting their time when they worry about "foundational crisis"; mathematicians consider that logicians are indulging in learned hair-splitting when they devote pages and volumes to discussing the meanings of meaning or the use of quotation marks and brackets.

The right course is to be as rigorous and detailed as the occasion or the purpose requires. But this is more easily said than done. For example, certain authors seem to dwell tirelessly on the obvious, while skipping the crucial and more difficult steps.

The matter of distinguishing expressions from that which is expressed may serve to illustrate some of the questions about rigour. There were occasions when failure to be careful about the distinction actually hindered greatly the advance of logic. It is now customary in logic and philosophy to stress the difference, usually using quotation marks to separate, for example, the city Peking from the word "Peking". At present, even those who do not want to spend much time on using the quotation marks rigorously, often find it necessary to declare, for example, "quotation marks are omitted in most cases since we believe that no confusion will arise from this negligence". Every now and then, we run into certain articles in which the authors are so meticulous about using quotation marks that it becomes very difficult to read and understand what is being said.

One might even distinguish logicians into two groups depending on whether or not they always try to use quotation marks consistently and exactly. It may be a matter of temperament. Or it may also be a question of whether one happens to be either too lazy or too busy.

5. *Approximation to intuition*

To put thoughts in words or to describe a particular experience involves formalization of intuition. It has been contended that no finite number of propositions could describe exhaustively all that is involved in a particular experience. In other words, it is impossible to formalize without residue the complete intuition at the moment.

The matter of approximating intuition by formalization is clearer with regard to mathematics. For example, we know intuitively many things about integers. If we are asked to characterize our notion of integers, one way of answering is to say that integers form a group with respect to addition, they form an ordered set with regard to the ordinary relation of being greater than, and so on. The notions of group, ordered set, etc., are more exactly defined or more formalized than the notion of integers. Consequently, such answers tend to clarify somewhat our notion of integers, but they are usually inadequate because they fail to characterize unambiguously the integers.

We may compare the place of abstract structures such as group, field, ordered set, etc., in mathematics with the place of general concepts in ordinary life. They all can be considered as results of formalization or abstraction which serve as tools of thinking and research. As tools they help to economize our thought, as is often remarked. For example, not only integers, but transformations in space, etc., all form groups; anything that we prove about groups in general, of course, applies also to the special groups which may differ from one another in many respects. Similarly, there are many different chairs which can all be employed to support buttocks. In this way formalization, closely tied up with abstraction, produces useful tools.

On the other hand, it is often hard to characterize adequately our intuition through the use of formal structures. For example, it is not easy to describe exactly the colour, shape, etc., of a particular chair. Peano's axioms are thought to be capable of characterizing completely our notion of positive integers. Yet, as Russell observed long ago, Peano's axioms are satisfied by all progressions such as the odd positive integers, the negative integers. Russell thought that only by calling in a set theory could we make a univocal characterization. More recent advances in logic show that he was wrong even in believing this.

In fact, as we know, there are important results which indicate unmistakably that we can formalize without residue neither the fundamental intuitive notion of positive integers nor the basic notion of sets or classes.

Thus, there is Gödel's famous theorem according to which, for any fairly rich system, we can find some property expressible in the system such that we can prove for each of the integers 1, 2, . . . that it has the property, but we cannot prove the general statement that all positive integers have the property in question. In other words, although intuitively if P(1) (*i.e.*, 1 has the property P), P(2), P(3), . . . are all true, then it must be the case that all positive integers have the property P; yet in no fairly strong logistic system can we formalize adequately this intuition so as to guarantee the performability of such an inference for all the properties P expressible in the system. It also follows that no ordinary axiom system can preclude the interpretation that besides the ordinary 1, 2, . . . the set of positive integers also contains certain other queer things; there is no way to formalize in an ordinary logistic system our intuition that 1, 2, . . . are the only integers.

On the other hand, there is no axiom system in which we can get *all* the real numbers or the classes of positive integers. This follows easily from Cantor's famous argument for non-denumerability. Thus, given any axiom system, we can enumerate all the classes of positive integers which can be proved to exist in the system, either by applying Löwenheim's theorem or by reflecting on the fact that the theorems of existence in the system can be enumerated. Hence, if we define with Cantor a class K of positive integers such that for each *n*, *n* belongs to K if and only if n does not belong to the nth class in the enumeration, then the existence of K cannot be proved in the system. In other words, although in the system we can also speak of all the classes of positive integers, we cannot really formalize without residue the intuitive notion of "all" with regard to classes of positive integers; in each formalized axiom system, there is always some class of positive integers that is left out.

6. *Application to philosophy*

The application of mathematical logic to the treatment of philosophical problems may also be viewed as an attempt to formalize. Such applications often give the impression that a formidable technical book expresses in tiresome exactitude more or less commonplace ideas which could be conveyed more easily and more directly in a few sentences of plain language. Yet, undoubtedly, there are cases where the appeal to formalization is of more than pedantic interest. For instance, Heyting's formalization of the intuitionistic view of logic and mathematics helps quite a bit in conveying Brouwer's ideas to those people who have a radically different orientation. Another example is the gradual formalization of the notion of being a definite property, employed for defining sets in Zermelo's axiomatic treatment of set theory.

Perhaps we can compare many of the attempts to formalize with the use of an airplane to visit a friend living in the same town. Unless you simply love the airplane ride and want to use the visit as an excuse for having a good time in the air, the procedure would be quite pointless and extremely inconvenient. Or we may compare the matter with constructing or using a huge computer solely to calculate the result of multiplying seven by eleven. When the problems are so simple, even the task of translating them into a language which, so to speak, the machine can understand would already take longer than if we were to calculate the results by memory or with a pencil and a sheet of paper.

It is a practical problem to decide what means of transportation to use in making a certain particular trip, or to decide whether it is feasible to build a computer to handle a certain given type of question. As we know, there are many different factors which are ordinarily taken into consideration before making the decision. Similarly, it is also a practical problem to decide in each particular case whether it is profitable to apply mathematical logic in handling a definite kind of problem.

The only difference is that the factors which have to be considered here are often more involved and less determinate.

Take the principle of verification. Various attempts at giving an exact definition of the notion of verifiability have failed. And systematic use of the logistic method has been recommended as the only way to a satisfactory solution. On the other hand, there is also the view that the important thing is a general attitude expressed vaguely in the rough principle of verification, rather than an exact definition of verifiability. Underlying this dispute, perhaps, are the varying attitudes toward the general desirability of crystallization of ideas.

This raises larger problems. Why should we want such crystallization in philosophy? What is the function and business of philosophy? Fortunately, general observations can be made without going into such hard questions.

7. *Too many digits*

After sketching an axiom system for his theory of probability, F. P. Ramsey goes on to say, "I have not worked out the mathematical logic of this in detail, because this would, I think, be rather like working out to seven places of decimals a result only valid to two". There are several disadvantages in working out a result to too many places. It uses up time which might be spent otherwise. It also makes the result harder to memorize or to include in future calculations, if anybody should want to make use of it. And pointless problems would arise regarding the last five places: do they exhibit any interesting pattern which would indicate the lawfulness of nature? Do they coincide with the five digits starting with the 101st in the decimal expansion of π? and so on.

How do we decide whether a result is valid only to two places? If the same experiment is repeated under different but, so far as we know, equally favourable circumstances, with results which agree satisfactorily only to the first two places, then we tend to conclude that the places after the second are not quite reliable. If most people refuse to calculate up to many places and a single person has an irresistible itch for reporting every result to at least seven places, it might be rather hard to decide whether his result is right.

The matter of constructing an exact theory of (say) probability contains an additional factor. Since ordinary language is not exact, new words are coined or ordinary words are given technical usage. In order to evaluate the theory, you have first to understand it. In order to understand it, you have first to learn a new language. Since it is usually impossible to explain clearly and exactly even the technical usages, a formal or exact theory can almost always be defended against charges that it does not conform to fact. As long as there is a sufficiently complicated system and a fairly big and energetic group of people who, for one reason or another, enjoy elaborating the system, we have a powerful school of learning, be it the theory of meaning, the sociology of knowledge, or the logic of induction. There is always the hope that further development of the theory will yield keys to old puzzles or fertilise the spirit of new invention. In any case, since there is mutual support between different parts of a given system, there is little danger that the discrepancy between one part and the facts should discredit the system. And of course if we are interested in the "foundations", there is no need to fear any immediate tests. The worst that can happen to such theories is not refutation but neglect.

8. *Ideal language*

Language is employed for expression and communication of thoughts. Failure in communication may either be caused by inadequate mastery of the language, or by internal deficiencies of the language: that is, if there is thought to be conveyed at all. Language is also sometimes used for talking nonsense. Here again, certain languages just seem to offer stronger temptations for doing so. And sometimes the language user is not careful enough, or he merely parrots others. In such cases he does not have thoughts or feelings to express, and there is, of course, no question of correct communication. A less serious disease is confused thinking, often involving internal inconsistency. This again is sometimes the fault of the language, such as the ambiguity of words and a misleading grammar.

The creation of an ideal language would yield a solution of these difficulties once for all. Such a language should be so rich, clear, and exact as to be sufficient both for expressing all thoughts and feelings with unmisunderstandable clarity, and for precluding nonsense. Given such a language, many problems now known as philosophical would be dissolved. Disagreement about what is to be taken as nonsense would lead to the construction of different ideal languages. There would be then the problem of understanding each other's ideal language.

An alternative to the ideal language is to handle each individual case separately and thoroughly. To explain at great length what we intend to say, to give concrete examples when possible, to invite questions and discussions. And to reflect carefully and ask what we really want to say, whether we do have something to say, whether we are not misled by false analogies or naive syntax.

The task of constructing a comprehensive ideal language is in many ways similar to that of finding a mechanical procedure to decide answers to all problems of mathematics. They are equally impossible. If and when these two tasks are clearly formulated, the impossibility can be proved definitely in both cases. In certain simple areas of logic and mathematics, we do possess decision procedures. Similarly in mathematical logic and theoretical physics we have more exact languages. But there is no mechanical method for finding decision procedures, and each significant mathematical problem calls for a special treatment. It is demonstrably impossible to reduce all mathematics to its decidable portion. It seems equally impossible to fit everything we say into the language of logic and physics. Moreover, these languages are more exact in their abstract set-up than in their actual use. It is a familiar experience that mathematicians who know the language of mathematics very well often offer fallacious proofs.

The quest for an ideal language is probably futile. The problem of formalization is rather to construct suitable artificial languages to meet individual problems.

9. *How artificial a language?*

The contrast between natural and artificial languages suggests a sharp distinction. Russian is natural, while Esperanto is artificial. But is the language of the biologists or that of the philosophers natural or artificial? Is Mr. Woodger's proposed language for biology natural or artificial? Hilbert's language for the Euclidean geometry is more exact and artificial than that of Euclid's *Elements*. So far as the development of human scientific activities is concerned, the creation of the language of the classical mechanics or of the axiomatic set theory was rather natural.

We might speak of degrees of artificiality, as perhaps measured by the amount of deviation from the natural course. The Chinese language spoken today differs to a rather great extent from that used two thousand years ago, although the changes have been mostly natural. If we had attempted two thousand years ago to bring about the same changes in one year's time, we would have had to create at that time a language quite artificial. To introduce an artificial language is to make a revolution. Unless there are compelling natural needs, the resistance will be strong and the proposal will fail. On the other hand, when an artificial language meets existing urgent problems, it will soon get generally accepted and be no longer considered artificial. Hence, it may be more to the point if we compare artificial languages with Utopian projects.

Attempts to formalize the theory of probability are sometimes criticized on the ground that the efforts fail to make contact with the crucial and burning problems of physical science. One ready reply is that the situation is the same with many interesting investigations in branches of mathematics such as abstract algebra, set theory, and topology. One may argue, however, that more new ideas and methods are introduced through such studies than through the researches on foundations of probability theory. Or maybe there is more substance behind the new languages of algebra and set theory and results obtained there are not as easily discredited by slight shifts of emphasis or subtle mistakes in the original analysis.

Mrs. Joan Robinson somewhere remarks that economists are usually behind their time. An urgent practical problem often ceases to be urgent or practical long before the discovery of a theoretically satisfactory solution. Whether it is worthwhile to continue the search for the solution of a problem which is no longer urgent depends to a large extent on whether the particular problem is intimately connected with larger issues, whether it is sufficiently intriguing intellectually, and whether it is likely to recur in the near future. Similarly, the value of an artificial language has to be decided in accordance with its elegance and its usefulness either in its direct applications or as a model to be followed in future constructions. In a certain sense, an interesting artificial language must not be excessively artificial.

10. *The paradoxes*

Much time and space has been devoted to the discussion of the logical paradoxes or contradictions. Sometimes it is said that these paradoxes bring to light the self-contradictory character of our logical intuition. Indeed, as we know, the formalization of logic and set theory was largely motivated by a desire to avoid the paradoxes and yet obtain what we ordinarily want.

It has been suggested that we take the paradoxes too seriously, largely because of our preoccupation with formalization and our lack of flexibility.

What is proposed instead seems to be this. Suppose we find a contradiction by a seemingly plausible argument. Since we get a contradiction, we see that the argument is really not correct and indeed must be faulty. So let us remember never to use the argument again. And that is the end of the matter.

However, when we say that the argument looks plausible, we mean, among other things, that each step of the argument also looks plausible. It seems necessary not only to reject the whole argument as a unit but to pin down exactly which step or steps in the argument caused the trouble. Hence, there are the various attempts to reject one or another of the steps as unwarranted. But why can we not say that although each step is in itself all right, they must not be combined in the particular way that leads to the contradiction ? Indeed, we may even use this possibility to justify the attitude of indifference, on the part of many working mathematicians, toward the paradoxes.

It is only when we come to constructing a formal system to embody our arguments that this procedure proves awkward. In a logistic system, we break up proofs and arguments into isolated steps so that if a step is valid at all, it is valid no matter where it occurs. In other words, certain combinations of shapes are taken as axioms so that they can be asserted as valid no matter where they occur ; and certain (finite) sequences of combinations of shapes are taken as justified by the rules of inference so that any such sequence, wherever it occurs, is taken as determining valid steps. For instance, if we agree to take as an axiom, for two specific sets named *a* and *b*, the assertion "Either *a* belongs to *b* or *a* does not belong to *b*", we can no longer reject the same statement as an unwarranted step when it occurs in an argument that leads to a contradiction.

Two alternatives to the customary logistic method are: (1) not to attempt any exact characterization of all the valid arguments of any important branch of mathematics ; (2) to list either all or samples of all the warranted and unwarranted whole specific arguments as inseparable units, instead of trying to break up all warranted arguments into a small number of basic atomic steps. The alternative (2) will either produce quite messy results or lead to something which is hardly distinguishable from a logistic system.

Irving Singer has read an earlier draft of the manuscript and corrected a number of sentences and phrases which were not idiomatic.

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